



## INDUCED CONSTRAINTS FOR STOCHASTIC OPTIMIZATION PROBLEMS

AMBI POLYCARP NAGWAI AND SANI SALIHU ABUBAKAR

Department of Statistics

Kano University of Science and Technology, Wudi. Nigeria.

### ARTICLE INFO

#### Keywords:

Stochastic Optimization,  
Sequential Decision, Induced  
Constraint

Correspondence author:

#### e-mail:

[hodcomputerscienceou@gmail.com](mailto:hodcomputerscienceou@gmail.com),  
[sanialihu14@gmail.com](mailto:sanialihu14@gmail.com)

#### Phone:

### ABSTRACT

*It is obvious that, decision must be taken every now and then, regarding the past, the present and the future. However, there is no decision without consequences, be it positive or negative. This paper is concerned with here and now decision, which is the decision we must take now and instantly. This is achieved by formulating a general framework for studying induced constraints thereby obtaining some very weak condition under which it is possible to pass to a somewhat more constructive description of these induced constraints.*

### INTRODUCTION

Stochastic optimization problems are essentially sequential decision problems. Given that at a particular stage a decision must be made one should take into account the fact that given the decision selected and given the possible disturbances which are due to random elements, the problem must remain a feasible problem. In other word the here and now decision must take into account the fact that we must satisfy the constraints to be encountered at the future stages of the problem whatever be the outcome of the random variables. Constraints generated by such future consideration are called induced constraints. In general, such constrains are hard to compute since deriving them involves a backing-up process. Moreover, for constrains involving random elements it is usually sufficient to satisfy them with probability 1 and not necessarily for all points of the distribution.

The purpose of this paper is to formulate a rather general framework for studying induced constraints and obtain some very weak condition under which it is possible to pass to a somewhat more constructive description of these induced constraints. We also give some indications how the general theory applies in the field of stochastic programming and stochastic optimal control.

#### Theory

We consider the following problem: let  $(\Xi, \mathfrak{F}, \mu)$  be a probability space where  $\mathfrak{F}$  is the  $\sigma$ -algebra and  $\mu$  is a probability measure. We shall assume that a topology compatible with  $\mathfrak{F}$  defined on  $\Xi$ . The random variable  $I$  denoted by  $\xi$  and point of  $\Xi$  by  $\xi$ . Let  $\mathfrak{x} \times \mathfrak{y}$  be a topological space such that every subspace of  $\mathfrak{x}$  is seperable in the induced topology, we shall say that  $\mathfrak{x}$  in hereditary separable.

Let  $f$  be a mapping from  $X \times Y \times \Xi$  into  $\mathbf{R}^m$  where  $X$  and  $Y$  are subset of  $\mathfrak{X}$  and  $\mathfrak{Y}$  respectively. The question is to characterize the set

$$D = \{x \mid \exists o \in f(x, y, \xi) \text{ a.s.}\}$$

Where a.s. stands for almost surely. The set  $D$  is obviously well defined in this manner; the goal is to find a somewhat more constructive description which would enable us to actually compute the set of acceptable  $X$ 's.

In term of our stochastic optimization problem we view the space  $\mathfrak{X}$  as representing the space from which a decision must be selected here and now. The space  $\mathfrak{Y}$  corresponds to the space of future decision whereas  $\xi$  stand for all future random events not yet observed. Thus finding the set  $D$  is to determine the set of acceptable here and now decisions which with probability 1 will enable us to find a feasible solution to the subsequent decision problem that we shall have to face.

We shall not consider the most general possible class of mapping  $f$ . Those of which we have been able to derive some result satisfying the following assumptions:

(A.1)  $f(x, y, \xi)$  is continuous in  $X$  (for fixed  $y$  and  $\xi$ ).

(A.2)  $f(x, y, \xi)$  is continuous in  $\xi$  (for fixed  $x$  and  $y$ ).

(A.3)  $f(x, y, \xi)$  is a upper semi continuous with respect to  $\xi$  for each fixed

$x$  in  $X$ , i.e. the graph of  $f(x, Y, \xi)$  is closed in the product space

$\Xi \times \mathbf{R}^m$  for each  $x$  in  $X$

(A.4)  $f(x, y, \xi)$  is continuous in  $x$  (for fixed  $y$  and  $\xi$ )

We define a relation between the point of  $X$  and  $\Xi$  as follows: we say that  $x$  is related to  $\xi$  which we write as

$$x \sim \xi, \text{ if } \exists o \in f(x, Y, \xi).$$

The set of all points in  $X$  related to a given  $\xi$  in  $\Xi$  is denoted by

$$\omega(\xi) = \{x \in X \mid x \sim \xi\}$$

And thus

$$\omega(\Xi) = \{x \in X \mid x \sim \xi \text{ for all } \xi \text{ in } \Xi\}$$

$$\Xi = \bigcap_{\xi \in \Xi} \{x \in X \mid x \sim \xi\}$$

Or more generally, for  $\Sigma$  an arbitrary subset of  $\Xi$

$$\omega(\Sigma) = \bigcap_{\xi \in \Sigma} \{x \in X \mid x \sim \xi\}$$

Similarly, we introduce the inverse map

$$\omega^{-1}(x) = \{\xi \in \Xi \mid x \sim \xi\}$$

and

$$\omega^{-1}(S) = \bigcap_{x \in S} \{\xi \in \Xi \mid x \sim \xi\}$$

For  $S$  an arbitrary subset of  $X$ . in order to avoid some pathological situations which are in little interest, we assume that

(A.4) For fixed  $x \in f(x, Y, \xi)$  is a measurable subset of  $\Xi \times \mathbf{R}^m$  where the measurable set are obtained as the  $\sigma$ -product of  $\mathfrak{Y}$ , and the borel algebra for  $\mathbf{R}^m$ . from this it follows immediately that

**Proposition 1.** For all  $x \in X$ , the set  $\omega(x)$  is measurable, i.e.

$$\omega^{-1}(x) \in \mathfrak{F}.$$

The mapping  $\omega$  and  $\omega^{-1}$  satisfy natural duality relations such as exhibited by the following proposition

**Proposition 2.** Suppose  $\omega^{-1}(S) \supset S$  and  $\omega^{-1}\omega(\Sigma) \supset \Sigma$  for  $S$  a subset of  $X$  and  $\Sigma$  a subset of  $\Xi$

**Proof:** suppose  $x \in S$ , then  $\omega^{-1}(x) = \{x \mid x \sim \xi\}$  and thus  $x \in \omega[\omega^{-1}(x)]$  since the  $\sim$  relation is reflexive. Note that this holds even if  $\omega^{-1}(x)$  is empty since  $\omega(\emptyset) = X$ . The other inclusion is obtained in the same manner.

From the properties of  $f$  we also get the following properties of  $\omega$  and  $\omega^{-1}$ .

**Proposition 3:** for every  $\xi$  in  $\Xi$ ,  $\omega(\xi)$  is a closed subset of  $X$  relative to the induced topology.

**Proof:** fix  $\xi$ , then  $x \in \omega(\xi)$  if and only if  $\exists o \in f(x, Y, \xi)$ . The proposition follows from (A.1) the continuity of  $f$  with respect to  $x$ .

**Proposition 4:** suppose  $\Sigma$  is a subset of  $\Xi$  such that the restriction of  $f(x, Y, \xi)$  is upper in semi continuous on  $\Sigma$ , then  $\omega^{-1}(x) \cap \Sigma$  is closed in  $\Sigma$  with respect to the induced topology.

**Proof:** this following immediately from (A.3) the closure of the graph of  $f(x, Y, \Sigma)$  in

$\Sigma \times \mathbf{R}^m$  and (A.2) the continuity of  $f(x, Y, \xi)$  with respect to  $\xi$ .

We can now reformulate the basic problem in terms of the map  $\omega$  and  $\omega^{-1}$  and ignore how they were generated.

**Problem.** Given a probability space  $(\Xi, \mathfrak{F}, \mu)$  with a topology on  $\Xi$  compatible with  $\mathfrak{F}$ , a map  $\omega$  defined on a subset  $X$  of an hereditarily separable space with a range into subset of  $\Xi$  and  $\omega^{-1}$  defined from  $\Xi$  into subsets of  $X$  satisfying:

(A.i)  $\Sigma \subset \Sigma'$  implies  $\omega(\Sigma) \supset \omega(\Sigma')$  and  $S \subset S'$  implies  $\omega^{-1}(S) \supset \omega^{-1}(S')$  in particular  $\omega^{-1}(\omega^{-1}(S)) \supset S$ ,  $\omega^{-1}(\omega(\Sigma)) \supset \Sigma$  for  $S \subset X$  and  $\Sigma \subset \Xi$

(prop. 2)

(A.ii) for all  $\xi \in \Xi$ ,  $\omega(\xi)$  is closed in  $X$  (prop. 3),

(A.iii)  $\omega^{-1}(x) \cap \Sigma$  is a measurable closed subset of  $\Sigma$  for all  $x$  in  $X$

(prop. 1 and 4),

Characteristic set

$$D = \{x \in X \mid \mu[\omega^{-1}(x)] = 1\}.$$

The following lemma is useful to establish the key theorem.

**Lemma.** Suppose  $\Sigma$  is a subset of  $\Xi$  of measure 1 then  $D \supset \omega(\Sigma)$ .

**Proof:**  $\{x\} \subset \omega(\Sigma)$  implies  $\omega^{-1}(x) \supset \omega^{-1}(\omega(\Sigma)) \supset \Sigma$  by (A.i) and thus  $x \in D$  since  $\mu(\omega^{-1}(X)) \geq [\mu(\Sigma)] = 1$  **Theorem 1.** There

always exists a subset  $\Sigma$  of measure 1 contained in  $\Xi$  such that  $\omega(\Sigma) = D$ . Moreover  $\Sigma$  can be selected to be  $\omega^{-1}(D)$

**Proof:** first assume that  $D$  is an empty then  $\omega^{-1}(D) = \Xi$  and  $\omega(\Xi) = \omega(\omega^{-1}(D)) \supset D = \phi$ , i.e.  $\Xi$  is the appropriate set. Now consider the case when  $D$  is nonempty. Take  $S$  a countable dense subset of  $D$ . The existence of such a set follows from the hereditary separability of  $X$ . Set  $\Sigma = \omega^{-1}(S)$ . If  $x \in S \subset D$  it follows from the definition of  $D$  that

$$\mu[\omega^{-1}(X)] = 1 \text{ thus } \mu(\Sigma) = \mu[\omega^{-1}(S)] = \mu[\bigcap_{x \in S} \omega^{-1}(X)] = 1$$

Since  $\Sigma$  can be written as the countable intersection of set of measure 1. We

now use the lemma from which it follows that since  $\Sigma \subset \Xi$  and  $\mu(\Sigma) = 1$  we have that  $\omega(\Sigma) \subset D$ . To obtain the other direction first observe that for all  $\xi$  in  $\Sigma$ ,  $\omega(\xi) \supset S$  but by (A.ii)  $\omega(\xi)$  is closed and consequently  $\omega(\xi) \supset D$  for all  $\xi$  in  $\Sigma$ . Thus  $\omega(\Sigma) \supset D$ .

**Corollary 1.** The set  $D$  is closed.

**Proof:** it is sufficient to observe that  $D = \omega(\Sigma)$  for some set  $\Sigma$  and that for all  $\xi$  in  $\Sigma$ ,  $\omega(\xi)$  is closed by (A.ii).

**Corollary 2.** If for all  $\xi$  in  $\Sigma$ ,  $\omega(\xi)$  is convex then  $D$  is a closed convex subset of  $X$ .

**Proof:** similar to corollary 1.

**Corollary 3.** If with its topology  $\Xi$  is a lindelöf space then the set  $D$  can be expressed as a countable intersection of the type  $\omega(\xi)$ . Moreover if  $\Xi$  is a discrete set, then it can be written as a finite intersection.

**Proof:** These characteristic of  $\Xi$  follow immediately from the definition of lindelöf in one case and the fact that  $\Sigma$  is a subset of  $\Xi$ .

It is worthwhile to observe that so far we have not made use of (A.iii) which in terms of the original problem followed from (A.3) and (A.4). But although we have been able to establish that there exist some set  $\Sigma$  such that  $D = \omega(\Sigma)$  we have no way to compute  $\Sigma$  except by knowing already  $D$  itself. The following theorem breaks this vicious circle. By  $\tilde{\Xi}$  we shall denote the support of  $\mu$ , i.e. the smallest closed subset of  $\Xi$  of measure 1. This set is usually available at the outset or can easily be obtained from the original data.

**Theorem 2.**  $D = \omega(\tilde{\Xi})$

**Proof:** Suppose  $x \in D$  then  $\mu[\omega^{-1}(x)] = 1$ . By (A.iii)  $\omega^{-1}(x) \cap \tilde{\Xi}$  is a closed set of measure 1. Since  $\omega^{-1}(x) \cap \tilde{\Xi}$  is closed in  $\tilde{\Xi}$  (A.iii) and  $\tilde{\Xi}$  is the smallest closed set of measure 1 it follows that  $\omega^{-1}(x) \supset \tilde{\Xi}$  for all  $x$  in  $D$ . By (A.i) we have that  $\omega(\omega^{-1}(x)) \subset \omega(\tilde{\Xi})$  for all  $x$  in  $D$  and also that  $D \subset \omega(\omega^{-1}(D)) \subset \omega(\tilde{\Xi})$ . To obtain the inclusion in the other direction it suffice to invoke once more the lemma, thus  $D = \omega(\tilde{\Xi})$ . Before we turn to some application we rephrase the last theorem in terms of the mapping  $f$ .

**Theorem 2':**  $D = \{x \mid \exists \xi \in \tilde{\Xi} \text{ s.t. } \omega(\xi) \ni x\}$  provided  $f$  satisfies assumptions (A.1), ..., (A.4).

### Application

a. Linear Stochastic Programs. We consider here the so-called stochastic program with recourse model. In view of the mathematical equivalence between stochastic programs with recourse and stochastic programs with chance constraints established in [7] it is rather straightforward to derive similar result for the later class of problems. The following constraints appear in such problems

$$TX + WY = 0 \quad Y \in K$$

Where  $x \in \mathbf{R}^n$  is the here and now decision, the matrices  $T$ ,  $W$  and  $p$  are random matrices whose element will be known only after  $x$  is selected but before the subsequent decision  $Y \in \mathbf{R}^m$  has to be chosen. The set  $K$  represents the amalgamation of the constraints imposed on  $Y$  by future decision and those which were explicitly present in the formulation of the problem.

We can string out the elements of  $T$ ,  $W$  and  $P$ , and we write them as one random variable

$$\underline{\xi} = (t_{11}, \dots, t_{m,n}, w_{11}, \dots, w_{m,n}, p_1, \dots, p_m).$$

We have thus this function

$$f(x, Y, \xi) = TX + WY$$

with  $X = \mathbf{R}^n$  and  $Y = K$ . it is obvious that this function satisfies (A.1) (A.2) and (A.4) provided  $K$  be any reasonable type of set. However, (A.3) is not satisfied in general. What we need to verify is that as  $\xi = (T, W, p)$  varies on  $\Xi$  and  $x$  is fixed, the multivalued map

$$T_\xi X + W_\xi(K) \quad p_\xi$$

is upper semi continuous as a function of  $\xi$  where

$$W_\xi(K) = \{t | t = W_\xi y, y \in K\}.$$

The only real task is to verify if  $W_\xi(K)$  is upper semicontinuous since for other terms there is no problem. If  $W_\xi = W$  is fixed and  $W(K)$  is closed then obvious (A.4) is satisfied and thus by theorem 2' we can obtain the constrains on  $x$  as an intersection of the form

$$D = \bigcap_{\xi \in \Xi} \{x | T_\xi x \in W(K)\}.$$

If  $K$  and  $\Xi$  are polyhedral so is  $D$  and one can derive the linear constraints induced on  $K$ . this is also the case when only  $p_\xi$  is random and the arbitrary distribution and  $K$  is polyhedral, see [9].

In general when  $W_\xi(K)$  depend on  $\xi$ , one has to study the properties of  $W_\xi(K)$ . The only work in this direction that we know of has been done when  $K$  belongs to certain classes of convex set see e.g. [2] [4] [11] for general result and [5] and [8] for result which lead us to a more constructive approach when  $K$  posses some specific properties.

b. Linear Discrete Stochastic Control Problem.

A discrete linear control problem can always be formulated as a multistage program with recourse. A general framework for the formulation of control problem with discrete observation is given in [6] and practical formulation can be found in [1] and [3] for example. We can thus apply the theory for multistage stochastic programs as derived in [10] where we see that to compute the induced constrains from one stage to the preceding one, the problem is identical to the one encountered from the linear stochastic program.

c. Nonlinear Stochastic programs.

When we use the term nonlinear, we refer to the constraints since if the constraints are linear and only the objective is nonlinear we can use the result for linear stochastic programs. We only consider the following problem

$$T(x) + W(y) \leq 0 \quad Y \in K$$

Where  $x \in \mathbf{R}^n$  is here and now decision;  $T$  and  $W$  are convex maps from  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively into  $\mathbf{R}^m$ , i.e.  $T_i(x) + W_i(y)$  are convex function in  $x$  and  $y$  for  $i=1, \dots, m$ ;  $y$  is the decision to be selected after observing  $\underline{\xi}$ . Again assumptions (A.1) (A.2) and (A.4) present no difficulties. To verify (A.4) it suffices to check if  $W(K)$  is a closed set. Obviously no general theory is available here and to derive specific condition we must rely on the specific properties of  $W$  and  $K$ . if in fact  $W(K)$  is closed we can here too derive the set  $D$  as an intersection of the same type than the one obtained for linear stochastic program, one should expect however that the task of obtaining explicit expression for the induced constrains set  $D$  might be much more complicated.

d. Stochastic optimal Control problem.

To extend this study to the stochastic control problem where observations occurs continuously it would be necessary to extend the theory to the situation where the range of the map  $f$  is infinite dimensional. The extension seems to present no difficulties but to obtain usable result, it seems that one should work with a different set of assumption since these seem very difficult to verify.

Conclusion: It is interesting to know that, Stochastic optimization problems can be of different categories, as presented in the paper, it could be either linear Stochastic, linear discrete, non-linear and Stochastic optimal control problem. The paper has solely deal on recourse model. Also, to be able to have optimal stochastic control where observations occur continuously, the theory must be

extended to a situation where the range of map of  $f$  is dimensionally infinite.

**Recommendation:**

- i. Further research be conducted to be able to obtain the extension of the theory.
- ii. Studies of this nature should encourage, for as long as we live decisions are bound to be taken whether now or for the future, which cannot be without consequences (favourable or unfavourable)

**REFERENCES.**

- R. S. Ellis and R. W. Rishel: An application of stochastic optimal control theory to the optimal rescheduling of airplanes. JACC conf. St. Louis (2001).
- J-L. Jolly: Une famille de topologie et de convergences sur l'ensemble des fonctionnelles convexes. Thesis (2000). Faculté des sciences de Grenoble.
- J. Midler: Investment in network expansion under uncertainty Rand Corporation report RM59,20PR, (2009)
- B. Van Custem: Element aléatoires á valeurs convexes Compactes. Thesis (1971). Université Scientifique et medicale de Grenoble.
- B. Van Custem: Random variables with convex compact values, application to stochastic programming (2001).
- R. Van Slyke and R. wets: Stochastic programs in abstract spaces, in stochastic Optimization and control, ed. F. karreman, wiley (2008), 25-45
- D. Walkup and R. Wets: Stochastic program with resource: Special forms proc. Princeton Symposium on mathematical programing, ed. H. Kuhn. Princeton press (2000), 139-161.
- D. Walkup and R. wets: Continuity of some convex valued Mapping, Proc. Amer. Math. Soc. 18(2007), 229-235.
- R. Wets: programming under uncertainty: the solution set, SIAM J. Appl. Math, 14(2001)1143-1151.
- R. Wets: Stochastic programs with recourse: A basic theorem for Multistage problem, Z. wahrscheinlichkeitstheorie view. Gebiete (2002)R. Wijsman: Convergence of Sequence of set, comes and function I, Bull, Amer. Math. Soc. 70(2004), 186-188. II. Trans. Amer. Math. Soc. 123(2006), 32-45