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## SOLUTION TO THE BRACHISTOCHRONE PROBLEM VIA FERMAT'S LEAST SQUARE PRINCIPLE

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### ABSTRACT

*This work studies the solution to the brachistochrone problem using the optical analogy of Fermat's least-time principle. In this analogy, the solution to the brachistochrone problem and the path followed by a ray of light is studied using Fermat's principle in optics. It was seen that the solution to the brachistochrone problem is equivalent to that of determining the path followed by a ray of light as it travels through a transparent medium of steadily decreasing density. Furthermore, the path followed by a ray of light and the brachistochrone problem yield cycloid. Interestingly, the fastest path between two points is a straight line only if the velocity is constant.*

### 1.0 Introduction

The history of the calculus of variations is tightly interwoven with the history of mathematics. The field has drawn the attention of a remarkable range of mathematical luminaries, beginning with Newton, then initiated as a subject in its own right by the Bernoulli family. The first major developments appeared in the work of Euler, Lagrange and Laplace. In the nineteenth century, Hamilton, Dirichlet and Hilbert are but a few of the outstanding contributors. In modern times, the calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics.

Many problems in physics have to do with extrema. Examples of such problems

include elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, and many, many others. When the problem involves finding a function that satisfies some extremum criterion, it may be approached using the various methods under the rubric of "calculus of variations". The basic approach is analogous with that of finding the extremum of a function in ordinary calculus (Ferguson, 2004).

The prototype problem introducing the calculus of variations is the "brachistochrone", from the Greek word meaning "shortest time" (Jeff and James, 2008). The statement of the problem supposes that a particle of mass  $m$  moves along some curve under the influence of gravity.

The motion is assumed to be in two dimensions and that the particle moves, starting at rest, from a fixed point  $a$  to fixed point  $b$ . It could be imagined that the particle is a bead that moves along a rigid wire without friction. The shape of the wire for which the time to get from  $a$  to  $b$  is minimized is what matters here (Erlichson, 1999).

This brachistochrone problem was first proposed by Bernoulli (one of them), and was solved by several people including Newton, though it's unlikely that he used the methods developed here, as the general structure we now call the calculus of variations was developed some decades later (Lebedev and Cloud, 2003).

Though the brachistochrone problem is considered to be the beginning of the calculus of variations (Benson, 1969; Bliss, 1925), (Boute, 2012) and a solution to the brachistochrone problem (Wan, 1995) would make use of general methods from branch of mathematics such as the Euler-Lagrange, and Jacobi tests, the Weierstrass excess function and more. Even so, many solutions that avoid the calculus of variations have been published (Hrusa and Troutman, 1981).

Different approaches to the brachistochrone problem have been developed, for example, Balder (2002), Brookfield (2006), and Lawler (1996). It is also worth mentioning that the minimization problem has been considered over larger classes of functions, in particular, absolutely continuous functions (Cesari, 1983).

Classical solutions to minimization problems in the calculus of variations are prescribed by boundary value problems involving certain types of differential equations, known as the associated Euler-Lagrange equations. The mathematical techniques that have been developed to handle such optimization problems are

fundamental in many areas of mathematics, physics, engineering, and other applications (Gemmer, 2006).

Fermat's principle states that the actual path between two points taken by a beam of light is the one which is traversed in the least time (Ross, 2009). In other words, light travels through the path in which it can reach the destination in least time. This principle of least time posits that an object follows a trajectory along which a particular integral of its kinetic and potential energy, called the time, is at an extremum (Ferguson, 2004).

The aim of this study is to highlight that the solution to the Brachistochrone problem is equivalent to that of determining the path followed by a ray of light as it travels through a transparent medium of steadily decreasing density. Since light accelerates as an object would if it were falling under gravity, it seems reasonable that the solution to the brachistochrone problem is isomorphic to the behavior of light (Struik, 1969).

The objectives of this study are to show how Euler-Lagrange equation can serve as a tool for finding the solution to the Brachistochrone problem, and show the relationship between path followed by a ray of light and the Brachistochrone Problem.

## **2.0 Comparison of the Optical and Mechanical derivations**

In this section, the path followed by a ray of light as it travels through a transparent medium of steadily decreasing density is studied using Fermat's principle in optics. The analysis of the study led to a differential equation whose solution is a smooth curve. The solution to the Brachistochrone problem is also found using calculus of variations. Comparison is made of these solutions and it is found that they both yield the same curve.

### 2.1 The optical analogy

The optical analogy describes the Brachistochrone problem in the light of Fermat's principle. The principle states that "light travels between two points along the path that requires the least time, as compared to other nearby paths". From Fermat's principle, one can derive a component of the law of reflection which states that the angle of incidence is equal

to the angle of reflection and that of the law of refraction (Snell's law).

To show these, consider a light ray traveling from point *A* to point *B* in two media with different indices of refraction, as shown in the Figure 1. The time to travel between the two points is the distance in each medium divided by the speed of light.

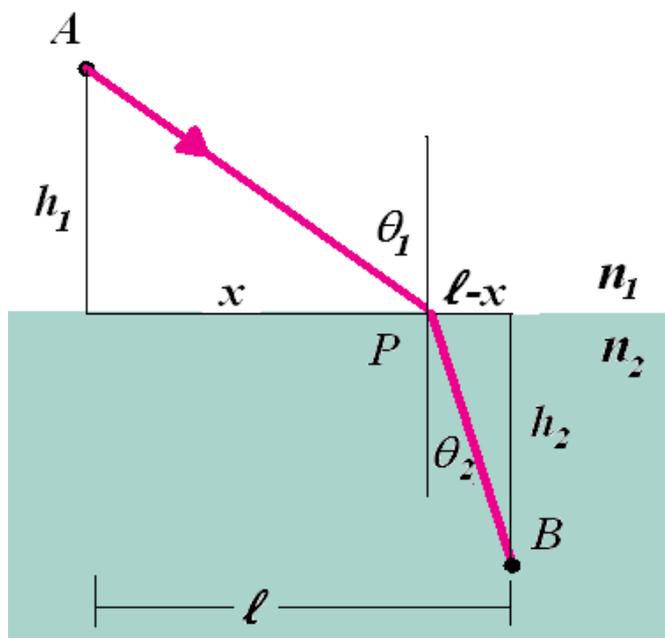


Figure 1

The fundamental law of refraction referred to as Snell's Law (Phillips, 1967) given by

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

(1)

where  $\theta_1$  is the angle of incidence and  $\theta_2$  is the angle of refraction. The  $n$ 's are the indices of refraction of the two media. The ratio of the speed of light in a vacuum ( $c$ ) to the speed of light in a solid ( $v$ ) is called the index of refraction ( $n$ ) of the solid denoted by

$$n = \frac{c}{v}$$

A representation for (1) below is of the form;

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2} = \frac{\sin \theta_2}{v_1} = c$$

(2)

The angle of incidence is defined as the angle between the incident ray of light and the normal, and the angle of refraction is defined as the angle between the refracted ray of light and the normal.

In general,

$$\frac{\sin \theta}{v} = c$$

(3)

where

$$\sin \theta = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \tag{4}$$

$$\frac{\sin \theta}{v} = \frac{1}{v \sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = c \tag{5}$$

The speed as a function of y is

$$v = (2gy)^{\frac{1}{2}} \tag{6}$$

Putting (6) in (5) then,

$$\sqrt{y + y \left(\frac{dy}{dx}\right)^2} = \frac{1}{c\sqrt{2g}} \tag{7}$$

and  $y + y \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{c\sqrt{2g}}\right)^2$

So that  $y + y \left(\frac{dy}{dx}\right)^2 = k$ , where  $k = \left(\frac{1}{c\sqrt{2g}}\right)^2$

Rearranging for  $dx$ , we have

$$dx = \frac{1}{2} \frac{kdy}{\sqrt{ky - y^2}} - \frac{1}{2} \frac{kdy + 2ydy}{\sqrt{ky - y^2}}$$

(8)

and upon integrating, one obtain that

$$x = k \sin^{-1} \sqrt{\frac{y}{k}} - \sqrt{ky - y^2} \tag{9}$$

Use the substitution  $y = k(\sin b)^2$  in (9). To show that this is a cycloid,

$$x = \frac{k}{2}(2b - \sin 2b), \quad y = \frac{k}{2}(1 - \cos 2b) \tag{10}$$

On putting  $\frac{k}{2} = R$  and  $2b = \theta$ , in equation (10), we have

$$x = R(\theta - \sin \theta) \text{ and } y = R(1 - \cos \theta).$$

### 2.2 The Brachistochrone problem

The Brachistochrone problem is that of finding the curve connecting two points (not on a vertical position), such that a particle sliding down this curve under gravity (in absence of resistance) from one point to another reaches in the shortest time (Aminu, 2015)

Now, assume a particle of mass  $m$  moving without friction along a path in constant gravity  $g$ . At speed  $v$ , the kinetic energy (K.E) is  $\frac{mv^2}{2}$ . As the particle

moves down over a level difference  $h$ , regardless of the shape of the path, gravity contributes an amount of energy  $mgh$ . Since there is no friction, this is fully imparted as an increase in kinetic energy. If the initial velocity is zero,

then  $\frac{mv^2}{2} = mgh$ . As a result, the velocity  $v$  at a level  $h$  below the starting point (not necessarily vertically) satisfies  $\frac{v^2}{2} = hg$ .

**2.2.1 Derivation of the solution**

Consider a frictionless wire connecting the two points A = (0, 0) and B = (1, 0). For notational convenience, assume that the positive y-axis extends below the x-axis, instead of above it, so that the point B is below A.

By the law of conservation of energy, K.E at B - K.E at A = potential energy at B

$$\frac{1}{2}mv^2 - 0 = mgy$$

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mgy$$

$$\frac{ds}{dt} = \sqrt{2gy}$$

The total time taken by the particle to move from A to B is given by

$$T = \int_0^1 dt = \int_0^1 \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_0^1 \frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_0^1 \frac{\sqrt{1+(y')^2}}{\sqrt{y}} dx$$

(11)

Solving equation (11) using Euler's equation:

$$F = \frac{\sqrt{1+(y')^2}}{\sqrt{y}} \tag{12}$$

Since the equation (12) is independent of x, i.e.,  $\frac{\partial F}{\partial x} = 0$ , it follows that

$$\frac{\partial F}{\partial y'} = \frac{1}{2\sqrt{y}} \frac{2y'}{\sqrt{1+(y')^2}} = \frac{y'}{\sqrt{y}(\sqrt{1+(y')^2})}$$

(13)

Considering the Beltrami Identity

$$c = F - y' \frac{\partial F}{\partial y'}$$

and substituting the values of F and  $\frac{\partial F}{\partial y'}$  into the Beltrami Identity, we obtain

$$\frac{\sqrt{1+(y')^2}}{\sqrt{y}} - y' \frac{y'}{\sqrt{y}(\sqrt{1+(y')^2})} = c$$

$$\frac{1+(y')^2 - (y')^2}{\sqrt{y}(\sqrt{1+(y')^2})} = c$$

$$1 = c(\sqrt{y+y(y')^2})$$

$$\frac{dy}{dx} = \sqrt{\frac{1}{c^2} - y}$$

Now, let  $\alpha = \frac{1}{c^2}$ , then

$$\int_0^x dx = \int_0^y \sqrt{\frac{y}{\alpha - y}} dy \tag{14}$$

Putting  $y = a(\sin b)^2 \Rightarrow dy = 2a \sin b \cos b db$  in (14) gives

$$x = \frac{\alpha}{2}(2b - \sin 2b), \quad y = a(\sin b)^2 = \frac{\alpha}{2}(1 - \cos 2b) \tag{15}$$

Substituting  $\frac{a}{2} = R$  and  $2b = \theta$  in (15), results in

$$x = R(\theta - \sin\theta) \text{ and } y = R(1 - \cos\theta)$$

This is the parametrization of a cycloid, with  $R$  being the radius of the disk and  $\theta$  the angle that the disk has rotated from its starting position.

### 3.0 Conclusion

It is seen that a brachistochrone is a segment of a cycloid arc. By a cycloid arc we mean a curve traced by a point on the rim of a disk that is rolled once along a line.

As proved earlier, it is seen here that the solution is a clever blend of physics and mathematics. Furthermore, the solution to the brachistochrone problem is equivalent to that of determining the path followed by a ray of light as it travels through transparent media of steadily decreasing densities.

The reasoning proceeds as follows. Begin by assuming that the light passes through a medium consisting of several layers, each with its own uniform density, which decreases as we move down through the layers. The ray passes from one layer to the next when its direction changes and its speed increase. It is noted that as the number of layers increases without bound, the path becomes a smooth curve.

It is interesting to note that the fastest path is not simply a straight line. If velocity

were constant, the least time path would indeed be straight. However, gravity accelerates the particle, and a steeper initial downward slope might allow gaining velocity faster.

In addition, Gemmer (2006) notes that Minimization problems that can be analyzed by the calculus of variations serve to characterize the equilibrium configurations of almost all continuous physical systems, ranging through elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, and many, many others. Many geometrical configurations, such as minimal surfaces, can be conveniently formulated as optimization problems. Moreover, numerical approximations to the equilibrium solutions of such boundary value problems are based on a nonlinear finite element approach that reduced the infinite-dimensional minimization problem to a finite-dimensional problem, to which the optimization techniques can be applied.

In this work, attempt has been made to scratch the surface of this wide ranging and lively area of both classical and contemporary research.

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