

## NUMERICAL ANALYSIS OF POINCARÉ MAPS IN 2-D CENTRAL AND HARMONIC POTENTIALS

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### ABSTRACT

*The trajectories of two dimensional central and harmonic potentials;*

*$V_{(x,y)} = (x^2 + y^2)^2$  and  $V_{(x,y)} = \frac{1}{2}(x^2 + y^2)$  respectively were integrated using the fourth*

*order Runge-Kutta algorithm which was incorporated as a subroutine (Appendix II) in a code designed by (Koonin and Meredith, 1990) and their respective Poincaré maps (plot of  $P_y$  Vs  $y$ ) were plotted. 4000 points were requested in each case, where at each point the input energy was conserved, which satisfies the condition for integrating a Hamiltonian system. The Poincaré maps in the two dimensional potential resembles two equal semi elliptical features, where as that of the harmonic potential represent two identical points. In all case they are found to be integrable and separable.*

**KEYWORDS:** *Poincaré maps, central and harmonic potential*

### INTRODUCTION

Henon-Heiles Hamiltonian is a two-dimensional time independent Hamiltonian system which was

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\left(x^2 + y^2 + 2x^2y + \frac{2}{3}y^3\right) \quad (1)$$

In this review a numerical study of the surprising behavior of the trajectories of a particle moving in two dimensions after rigorous integrations will be analysed in some specific forms of the potential. An overview of the study of these systems can be found in (Helleman, 1980), (Abraham and Marsden, 1978) and (Rice, 1980). The modifications of these systems can also be retrieved from (Tally and Muttat, 2004).

$$H = \frac{1}{2}[P_x^2 + P_y^2] + V(x, y) \quad (2)$$

Where  $\frac{P_x^2}{2}$  and  $\frac{P_y^2}{2}$  are the kinetic energy of the unit mass particle and  $V$  is the potential energy of the unit mass particle in two-dimension. Given some specified initial values of

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = p_x; \quad \frac{dy}{dt} = \frac{\partial H}{\partial p_y} = p_y \quad (3)$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}; \quad \frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y}$$

For any potential  $V$ , these equation conserves the energy,  $E$  so that the constraint

$$H(x, y, p_x, p_y) = E \quad (4)$$

restricts the trajectory to lie in a three-dimensional manifold embedded in a four-dimensional phase space. Apart from these, very

originated as a model in galactic dynamics (Henon and Helies, 1964). The equation of this Hamiltonian is given by

The interest in this model arises from the fact that it is connected to a physical problem and also in spite of its simplicity it presents a rather rich complex dynamics.

Let us consider a particle of unit mass moving in a potential  $V$  in two dimensions and assume that  $V$  is such that the particle remains confined for all times if its energy is low enough. If the momenta conjugate to the two coordinates  $(x, y)$  are  $(P_x, P_y)$  then the Hamiltonian takes the form.

the coordinates and momenta, the particles trajectory is represented by their time evolution which is governed by the four coupled first order differential equation (Hamilton's equation) as; -

few other general statements can be made about the evolution of the system.

The fundamental aspect of the two dimensional Hamiltonians that require some relevant statement on the trajectories can be made are those that are integrable. For this class of potentials, there is a second function of the coordinates and momenta, apart from the  $V(x,y)=V_x(x)+V_y(y)$ ,

Where  $V_{x,y}$  are two independent functions, so that the Hamiltonian separates in to two parts, each involving only one coordinate and its conjugate momentum,

$$H = H_x + H_y; H_{x,y} = \frac{1}{2} P_{x,y}^2 + V_{x,y} \tag{6}$$

The motion of x and y therefore decoupled from each other and each of the Hamiltonian  $H_{x,y}$  is separately a constant of motion. In the case of central potential it can be defined as;

$$V(x,y) = V(r); r = (x^2 + y^2)^{\frac{1}{2}}, \tag{7}$$

**Fourth Order Runge-Kutta method**

A precise computational method is required to accurately and quickly perform an orbit simulation. Simple methods (e.g. Euler, modified Euler) need a very small time step, and thus a large amount of computing time is needed to keep it stable. Therefore, a more elaborate

$$\frac{dy}{dx} = f(x,y) \tag{8}$$

method is necessary to increase accuracy and to reduce calculation time. A fourth order Runge-Kutta is very well situated for this purpose, as it is stable at large time step, accurate and relatively fast. The Runge-Kutta method of integrating ODE of the form

is a step by step process of obtaining an approximation for  $y_{i+1}$  (Henon, M. 1982) from the value  $y_i$ . Among its advantage are, there are no functions other than  $f$  that are needed, no subsidiary differentiation is needed and no subsidiary starting values need to be calculated.

The basis of this method is to simulate the Taylor series for  $y(x_i + h)$  (Gear, 1971), not by calculating all the higher derivatives of  $y$  at point  $x$ , but by taking a particular combination of the values of the first derivatives of  $y$  evaluated at a number of carefully chosen point. Equation (3) is used to evaluate the first derivatives, the accuracy of this simulation can be made to be up to whatever power of  $y$  is desired, but naturally the greater the accuracy the more complex the calculation and in any case rounding errors cannot be ultimately be avoided.

**Algorithm**

The program EX 2 (Konin and Meredith, 1990) which is a multi subroutine code integrates trajectories in the Hamilton's equation (3) and finds the corresponding surface of section (SOS). Subroutine ARCHON created the surface for one value of the energy by calling subroutine SOS

energy, that is a constant of the motion; thus the trajectory is now constrained to a two dimensional manifold of phase space. Two famous kinds of integrable systems are separable and central potentials. In the separable case;

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once for each set of initial conditions requested. Subroutine SOS prompts for initial values of  $y$  and  $p_y$  calculates  $P_y$  from energy conservation (recall  $x=0$  or the surface of section) and execute the time integration (subroutine STEP) until the requested number of surface of section points have been found. Subroutine step takes a time step using the fourth-order Runge-Kutta step (subroutine RUNGE); the derivatives of the coordinates and the momenta are calculated by subroutine EVAL and functions XDERIV and YDERIV. These routines also check for crossing of trajectories with the y-axis (i.e. whenever  $x=0$ ). If a crossing is found, it temporarily switches to  $t$  as the independent variable and integrates backward to find the surface of section intersection.

The form of the potential may be changed by editing the potential  $V$  and its derivatives XDERIV and YDERIV, which are located in FUNCTION  $V$  (Appendix 1). A change also could be affected in the graphic limits for the trajectory in subroutine TRJINT and should review values DYO and TOLY in (subroutine INIT) used in searching the limits on the energy (also subroutine INIT) should be the smallest and largest values of energy that give rise to bounded motion.

**Input**

The physical parameter entered from the main menu is the energy [0.1], the initial condition for  $y[0.0]$  and  $p_y[0.0]$  are requested from the subroutine SOS so that more than one set of initial conditions can be used in creating a single surface of section. Limits on  $y$  depend only on the energy and are calculated by subroutine LIMITS; limits of  $P_y$  depend on the energy and  $y$ , and are calculated in subroutine SOS. The numerical parameters are time step [0.12] and number of surface of section points [4000].

**Output**

The text output (results output) at each surface of section intersection include the time, kinetic

$$V_{(x,y)} = (x^2 + y^2)^2 \tag{9}$$

$$= x^4 + 2x^2y^2 + y^4$$

Therefore

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(x^4 + 2x^2y^2 + y^4)$$

$$= 4x^3 + 4xy^2 \tag{10}$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(x^4 + 2x^2y^2 + y^4)$$

$$= 4y^3 + 4yx^2 \tag{11}$$

Equation (9), (10) and (11) are therefore replaced conveniently in the code, in Real function  $V(x, y)$  as

```
V = (x**2 + y**2)**2
XDERIV = 4 * x**3 + 4 * x * y**2
YDERIV = 4 * y**3 + 4 * y * x**2
```

The values of  $y$ ,  $P_y$  and  $E$  are retained from the default values but the number of Poincare map points was changed to 4000 from the default value of 100. By making the appropriate modifications in the code, data's were accumulated in a text file with extension.grf while the graphic output (SOS) was stored in a graphic file with extension .psg.

**Harmonic potential (separable)**

The Henon-Heiles potential in the code was changed to a harmonic potential i.e.

$$V = \frac{1}{2}(x^2 + y^2) \tag{12}$$

therefore  $\frac{\partial V}{\partial x} = x$  (13)

similarly

$$\frac{\partial V}{\partial y} = y \tag{14}$$

Equations (12) (13) and (14), were replaced conveniently in the code in routine function

$V_{(x,y)}$  as;

energy, total energy and percentage change in energy since the integration began. The graphics output is the trajectory and the surface of section points (output when the requested number of SOS points have been determined). The surfaces of section points (Poincare maps) for the central and harmonic potentials were plotted respectively, allowing the creation of sections similar to those in fig 1 and 2.

**RESULTS AND DISCUSSION**

**Central potential**

The Henon-Heiles potential in the code was changed to a central i.e.

$$V = \frac{1}{2}(x^{**2} + y^{**2})$$

$$XDERIV = x$$

$$YDERIV = y$$

The initial value of  $y = 0.05$ ,  $P_y = 0.05$ ,  $E = \frac{1}{12} = 0.083eV$

The number of SOS points = 4000.

By making the appropriate modifications in the code, data's were accumulated in the respective files as in the case of central potential.

**CONCLUSION**

The surface of section (Poincare map) for the central and harmonic potentials were constructed using the famous fourth order Runge-Kutta algorithm after integrating the Hamilton's equation and making appropriate modifications in the standard Henon-Helies potential, and it

was found to be integrable in all cases. The Poincare maps in the case of central potential was found to be similar to two equi-elliptical structures and the one of harmonic potential are two distinct and repeating points one on the positive axis and the other on the negative axis of the map.

**(SOS) SHOWING THE HARMONIC AND CENTRAL POTENTIAL.**

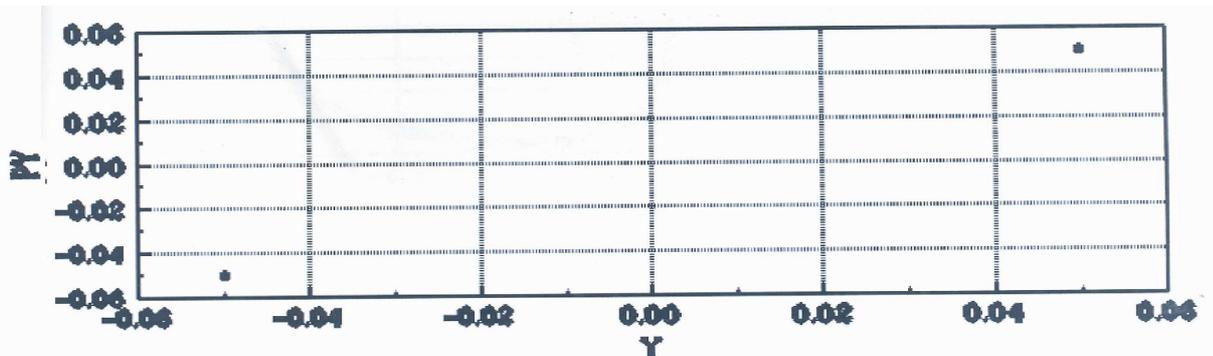


Figure 1: harmonic potential

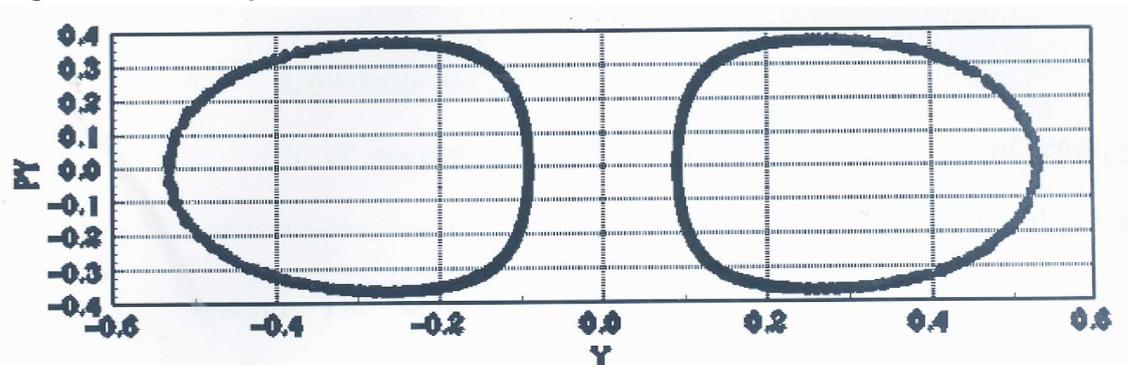


Figure 2: The central potential

**Appendix 1**

The Henon Helies potential code before making appropriate changes to represent the Central and harmonic potentials

```

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
  REAL FUNCTION V(X,Y)
C Calculates the potential and forces
C
C If you change the potential, you may also need to change DY0 and TOLY
C for Y limit searches, as well as limits for X and Y in TRJINT
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C Passed variables:
  REAL X,Y          !coordinates
C Functions:
  REAL XDERIV,YDERIV  !x and y derivatives of the potential
    
```



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